연구논문집 제4권 제2호 p.95~99 1979

超限基數들에 關한 考察

朴 榮 植

敎 養 科

<要 約>

本 論文은 超限基數들(또는 無限集合의 基數들)에 關한 것이다. 論文의 目的은 有限集合의 基數들의 性質을 알고 超限基數들($\aleph_0, c, 2^{\aleph_0}, 2^c, \aleph_0^{\aleph_0}, \aleph_0^c, c^{\aleph_0}, c^c$)의 一般的 特性을 조사하여 그들간의 相互關係를 硏究하는데 있다.

A Note on Transfinite Cardinal Numbers

Park, Yong Sik

Dept. of Basic Studies

<Abstract>

This paper is concerned with transfinite cardinal numbers(or the cardinal number of infinite sets). The Purpose is to investigate the general properties of transfinite cardinal numbers ($\aleph_0$, $c, 2^{\aleph_0}, 2^c, \aleph_0^{\aleph_0}, \aleph_0^c, c^{\aleph_0}, c^c$) and study their correlations as follows:

1. Algebraic numbers and $\aleph_0$.
2. Other sets with cardinality c and $\aleph_0$.
3. Cardinal exponents and $\aleph_0, c$.
4. The cardinality of the set of function and subset.

Cardinalities are to be well-defined sets, and every set is to be equipotent with exactly one cardinal number. The cardinal number of any denumerable set is customarily designated by the symbol $\aleph$. (aleph-null). It is useftl to distinguish between finite cardinal numbers-that is, cardinal numbers of finite sets-and infinite, or transfinite cardinal numbers, which are the cardinal number of infinite sets. We shall show to investigate a general definition of cardinal numbers of infinite sets, or transfinite cardinal numbers.

I. The general definitions of the cardinality of the set.

Definition: Let A and B be finite sets and $\#(A)$ be the cardinal number of the set A. Then $\#(A)=\#(B)$ implies that there exists a bijection $f: A\longrightarrow B$ (or a bijection $g: B\rightarrow A$). By Canter the above definition is applied to infinite sets. [4]

Definition 2 : Let α,β be cardinal numbers and A, B be sets such that $\#(A)=\alpha$, $\#(B)=\beta$, We define $\alpha\leq\beta$ by $\alpha\leq\beta \Longleftrightarrow \exists$ an injection $f: A\longrightarrow B$. [4]

(The properties of cardinal numbers of the finite sets)

Let α,β and γ be cardinal numbers.

(1) Transitivity of inequality; $\alpha\leq\beta$ and $\beta\leq\gamma \Rightarrow \alpha\leq\gamma$.

(2) Commutativity of addition and multipli-

cation; $\alpha+\beta=\beta+\alpha$, $\alpha\beta=\beta\alpha$.

(3) Associativity of addition and multiplication; $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$, $(\alpha\beta)\gamma=\alpha(\beta\gamma)$.

(4) Invariance of inequality with respect to addition and multiplication;
$\alpha\leq\beta\Rightarrow\alpha+\gamma\leq\beta+\gamma$, $\alpha\gamma\leq\beta\gamma$ for every γ.

(5) $\alpha\leq\beta\Longleftrightarrow\exists\gamma$ s. t. $\alpha+\gamma=\beta$.

(6) $\alpha<\beta\Longleftrightarrow\alpha\leq\beta$ and $\alpha\neq\beta$. [3]

Definition 3 : If a set A is equivalent to N, the set of Natural numbers, then A is called denumerable and is said to have cardinality $\aleph$. (read aleph-null or aleph naught), $\#(A)=\#(N)=\aleph_0$. [5]

Definition 4 : Let A be equivalent to the interval [0, 1] or R, the set of real numbers, then A is said to have cardinality c and to have the power of the continuum, $\#([0,1])=\#(R)=\#(A)=C$. [5]

Definition 5 : Set A is equivalent to set B, denoted by $A\sim B$ if there exists a function $f: A\longrightarrow B$ which is both 1—1 and onto. The function f is then said to define a 1—1 correspondence between sets A and B. [2]

Ⅱ. Algebraic Numbers and $\aleph_0$

The set of real algebraic numbers contains all rational numbers and all real roots of these rational numbers. We shall show that it must be a likely candidate for an infinite set with cardinal number greater than $\aleph_0$.

Theorem 1 : There is a 1—1 correspondence between the algebraic numbers and the positive integers. Hence the set of the algebraic numbers has cardinal number $\aleph_0$.

Proof: By a real algebraic number is meant any real root of an equation of the form, $a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=0\cdots(*)$ where the a's are all integers with a_n not zero, and where the n's are positive integers. Let h be height of(*). $h=n+a_n+|a_{n-1}|+\cdots\cdots+|a_1|+|a_0|$. To each height h there corresponds a finite number of polynomials, hence a finite number of algebraic numbers. We know that no polynomial equation of degree n can have more than n distinct roots. For any specified height h, there are only a finite number of polynomials having this designated height. For instance, we can find the polynomial with $h=3$, $h=4$, and $h=5$ respectively.

Polynomials with height 3 : $x^2, 2x, x+1, x-1, 3$.

Polynomials with height 4 : $x^3, 2x^2, x^2+x, x^2-x, x^2+1, x^2-1, 3x, 2x+1, 2x-1, x+2, x-2, 4$.

Polynomials with height 5 : $x^4, 2x^3, x^3+x^2, x^3-x^2, x^3+x, x^3-x, x^3+1, x^3-1, 3x^2, 2x^2+x, 2x^2-x, 2x^2+1, 2x^2-1, x^2+x+1, x^2-x+1, x^2+x-1, x^2-x-1, x^2+2x, x^2-2x, x^2+2, x^2-2, 4x, 3x+1, 3x-1, 2x+2, 3x-2, x+3, x-3, 5$.

Accordingly, we can begin our customary listing for the algebraic numbers $\{S_1, S_2, S_3, \cdots, S_n, \cdots\}$ where $S_1=0$(the root of polynomial of height $h=2$), $S_2=-1$, $S_3=1$ ($h=3$), $S_4=-2$, $S_5=-\frac{1}{2}$, $S_6=\frac{1}{2}$, $S_7=2$ ($h=4$), $\cdots$

If we continue to list only those roots which have not appeared before as we progress up ward to new heights along the scale of positive integers, we shall have a sequence of distinct algebraic numbers. Since every polynomial must have some positive integer as heighs, the set of the algebraic numbers, $\{S_1, S_2, S_3, \cdots, S_n, \cdots\}$ is equivalent to the set of positive integers. Hence the set of the algebraic number has cardinality $\aleph_0$.

Ⅲ. Other sets with cardinal numbers c and $\aleph_0$.

Having established the fact that the set of all points(real numbers) in the interval [0, 1] has cardinality c, We shall now track down other sets with this same cardinal number. Also we shall investigate the relation of cardinality $\aleph_0$ and c.

Theorem 2 : (1) Every open interval has cardinality c.

(2) $\#(I\times I)=\#(I)$, where I denotes the open interval $(0,1)$.

Proof: (1) Let $f:(a,b)\longrightarrow R$ defined by $f(x)=\frac{|y|}{b-|y|}$ for all $y\epsilon(a,b)$, then f is a bijection and thus $\#((a,b))=\#(R)=C$

(2) Let $f:I\longrightarrow I\times I$ defined by $f(x)=\left(x,\frac{1}{n}\right)$ for all $n\epsilon N$ and $x\epsilon(0,1)$, then f is injective and whence $\#(I)\leq\#(I\times I)$. Next, we shall find an injection $g:I\times I\longrightarrow I$ defined by $g\{(x,y)\}=z$ for every $(x,y)\epsilon I\times I$. Let $x=0.a_1a_2\cdots a_n\cdots(0\leq a_n\leq 9)$ and $y=0.b_1b_2\cdots b_n\cdots(0\leq b_n\leq 9)$. Then we shall define the number $z=0.a_1b_2a_2b_1\cdots a_nb_{n+1}a_{n+1}b_n\cdots$. If $(x_1,y_1)\neq(x_2,y_2)$, then $x_1\neq_2$ or $y_1\neq y_2$. Therefore the expansion of $z_1=g\{(x_1,y_1)\}=0.a_1{}^{(1)}b_2{}^{(1)}a_2{}^{(1)}b_1{}^{(1)}\cdots a_n{}^{(1)}b_{n+1}{}^{(1)}a_{n+1}{}^{(1)}b_n{}^{(1)}\cdots$ will be different from of $z_2=g\{(x_2,y_2)\}=0.a_1{}^{(2)}b_2{}^{(2)}a_2{}^{(2)}b_1{}^{(2)}\cdots a_n{}^{(2)}b_{n+1}{}^{(2)}a_{n+1}{}^{(2)}b_n{}^{(2)},\cdots$, where $0\leq a_n{}^{(1)}, a_n{}^{(2)}, b_n{}^{(1)}, b_n{}^{(2)}\leq 9$. This implies $z_1\neq z_2$. Therefore if $x_1\neq x_2\Rightarrow g\{(x_1,y_1)\}\neq g\{(x_2y_2)\}$ or if $y_1\neq y_2\Rightarrow g\{(x_1,y_1)\}\neq g\{(x_2,y_2)\}$, and thus g is injective and whence $\#(I\times I)\leq\#(I)$. By the cantor-Bernstein-Schröder theorem $\#(I\times I)=\#(I)=C$

From theorem2 the following Lemma 1 can be proved.

Lemma 1 : (1) Any two open intervals have the same cardinal number c.

(2) The cardinal number of $(0,1)$, $(0,1]$, $[0,1)$, and $[0,1]$ all have cardinality c.

Theorem 3 : Let N be the set of natural numbers and R be the set of real numbers. Then $\#(N)=\aleph_0<\#(R)=\text{c}$.

Proof: Since we have $N\subset R$, hence $\#(N)\leq\#(R)$, that is, $\aleph_0\leq\text{c}$ and $\aleph_0\neq\text{c}$. Then there does not exist a bijection $f:N\longrightarrow R$. Thus $\#(N)<\#(R)$, i.e., $\aleph_0<\text{c}$.

Lemma 2 : (1) $\#(N\times N)=\aleph_0$,

(2) $\aleph_0+\aleph_0=\aleph_0$,

(3) If $n\epsilon N$, then $n+\aleph_0=\aleph_0$, $n\aleph_0=\aleph_0$, $\aleph_0\text{c}=\text{c}$, and $n\text{c}=\text{c}$

Proof: (1), (2) are trivial.

(3) We have $\#(\{1,2,3,\cdots,n\})=n$ and $\#(\{n+1,n+2,\cdots\})=\aleph$.

Since $\{1,2,3,\cdots,n\}\cap\{n+1,n+2,\cdots\}=\phi$, then $\#(\{1,2,\cdots n\})+\#(\{n+1,n+2,\cdots\})=\#(\{1,2,\cdots,n,n+1,n+2,\cdots\})=\aleph_0=\#(N)$.

Next, since $\text{c}\leq\text{c}n\leq\text{c}\aleph_0\leq\text{c}\cdot\text{c}=\text{c}$ and $\aleph_0\leq\aleph_0 n\leq\aleph_0\aleph_0=\aleph_0$,

hence $\text{c}n=\text{c}\aleph_0=\text{c}$ and $\aleph_0 n=\aleph_0$.

From the above statements we can know as follows:

$(\text{c}+\text{c})+\text{c}=\text{c}+(\text{c}+\text{c})=\text{c}+\text{c}=\text{c}$, $(\aleph_0+\aleph_0)+\aleph_0=\aleph_0+\aleph_0=\aleph_0$(commutativity of addition).

Definition 6 : (1) The cardinal number α is infinite$\Longleftrightarrow\aleph_0\leq\alpha$

(2) A set A is infinite if $\aleph_0\leq\#(A)$, that is, $\exists$ an iniection $f:N\longrightarrow A$. [4]

Theorem 4 : If the cardinal number α is infinite, then $\alpha+\aleph_0=\alpha$

Proof: Since α is infinite, $\aleph_0\leq\alpha$. By the properties of cardinal number of the finite sets, $\exists\gamma$(cardinality) s.t. $\gamma+\aleph_0=\alpha$. Thus $\alpha+\aleph_0=(\gamma+\aleph_0)+\aleph_0=\gamma+(\aleph_0+\aleph_0)=\gamma+\aleph_0=(\alpha-\aleph_0)+\aleph_0=\alpha$.

We can prove easily the following Lemma 3.

Lemma 3 : (1) If $a<b$, then the closed interval $[a,b]$ has cardinal number c.

(2) If cardinal number α is infinite, then $\alpha+\text{n}=\alpha$ for every $n\epsilon N$.

Proof: (1) Since $(a,b)\cap\{a,b\}=\phi$, we have $\#([a,b])=\#((a,b))+\#(\{a,b\})=2+\text{c}=\text{c}$

(2) By theorem4 $\alpha+\aleph_0=\alpha$, hence $\alpha+n$

$=(\alpha+\aleph_0)+n$

$=\alpha+(\aleph_0+n)$

$=\alpha+\aleph_0$

$=\alpha$

Definition 7 : If α,β are any cardinal numbers and A,B are sets such that $\#(A)=\alpha$, $\#(B)=\beta$, then we define $\alpha\beta$ by $\alpha\beta=\#(A\times B)$ [3]

Lemma 4 : (1) $\#(N\times N\times N\times\cdots\times N)=\aleph_0$

(2) $\#(I\times I\times I\times\cdots\times I)=\text{c}$

Proof: (1) $\#(N\times N\times\cdots\times N)=\#(N)\#(N)\cdots\#(N)=\aleph_0\aleph_0\cdots\aleph_0$

$=\aleph_0$

(2) Let I donote the open interval $(0,1)$ and

$\#((0,1))=c$ Then we have $\#(I\times I\times\cdots\times I)$
$=\#(I)\#(I)\cdots\#(I)$
$=c\cdot c\cdots c$
$=c$

Ⅳ. Cardinal exponents($m^n, 2^{\aleph_0}, 2^C, \aleph_0^{\aleph_0}, \aleph_0^C, c^n$) and $\aleph_0$, c.

There may be a definite connection between functions and exponentiation. The exponent represents the cardinal number of the so-called domain of the function, the base represents the cardinal number of the range.

Definition 8 : Let S be a set with cardinal number n and let T be a set with cardinal number m. Then m^n is the cardinal number of the set of all possible functions on S with values in T, that is, the elements of the new set under consideration are functions, the exponent is the cardinal number of the set on which the function is applied and the base is the cardinal number of the set of possible values which the function assume. [5]

Example: Let S be the set $\{a,b,c\}$ and $T=\{0,1\}$. Then the set of functions on S on T will have cardinal number 2^3. A sampling of these functions is as follows:

a	0	0	1	1	1	0	1	0
b	1	0	0	1	0	1	1	0
c	0	1	1	0	0	1	1	0

We will define the functions set as the set of all triples with the number 0 and 1, that is, (0,1,0), (0,0,1), (1,0,1), (1,1,0), (1,0,0), (0,1,1), (1,1,1), (0,0,0).

Definition9: If A, B are any sets, then we denote by B^A the set defined by $^\forall f: f\epsilon B^A \Longleftrightarrow f: A\longrightarrow B$. In other words, B^A is the set of all graphs of mappings from A to B. We know that $\#(B^A)=\#(B)^{\#(A)}$ when A and B are finite [5]

Definition 10 : $B(A)$ is the set of all subsets of A.

$^\forall B: B\epsilon B(A)\Longleftrightarrow B\subset A$ [4]

Theorem 5 : The set of all subsets of N has cardinality c, i.e., $2^{\aleph_0}=c$

Proof: Let $\#(\{0,1\}^N)=2^{\aleph_0}$ and $\#(I)=c$ and $\varphi:\{0,1\}^N\longrightarrow I$ defined by $\varphi(f)=0.(f(1)+2)(f(2)+2)\cdots(f(n)+2)$, where $f\epsilon\{0,1\}^N$ and I denotes open interval (0,1). $\varphi(f)$ is a proper expansion, since each digit is either 2 or 3, moreover $\varphi(f)\epsilon\left[\frac{2}{9}, \frac{3}{9}\right]\subset I$. If $f_1\neq f_2$, then $f_1(n)\neq f_2(n)$ for some n and $f_1, f_2\epsilon\{0,1\}^N$. Thus $\varphi(f_1)\neq\varphi(f_2)$, since $\varphi(f_1)$ and $\varphi(f_2)$ have distinct proper decimal expansions. Therefore φ is injective and $\#(\{0,1\}^N)\leq\#(I)$, i.e., $2^{\aleph_0}\leq c$. Next, to prove an injection $\phi: I\longrightarrow\{0,1\}^N$ defined by ϕ_y (instead of the usual $\phi(y)$ for every $y\epsilon I$.)

Let us consider the binary expansion of y, $y=0.\phi_y(a_0)\phi_y(a_1)\phi_y(a_2)\cdots\cdots\phi_y(a_n)\cdots$, ($\phi_y(n)=0$ or 1) assigned by $\phi_y\epsilon\{0,1\}^N$ such that $\phi_y: N\longrightarrow\{0,1\}$ for every $a_n\epsilon N$. Then we can write $y=(\phi_y(a_0), \phi_y(a_1), \phi_y(a_2), \cdots, \phi_y(a_n)\cdots)$.

Define $y_0=(1, \phi_y(a_1), \phi_y(a_2), \phi_y(a_3)\cdots\phi_y(a_n), \cdots)$,
$y_1=(\phi_y(a_0), 1, \phi_y(a_2)\cdots, \phi_y(a_n), \cdots)$,
$y_2=(\phi_y(a_0), \phi_y(a_1), 1, \phi_y(a_3)\cdots, \phi_y(a_n), \cdots), \cdots$
$y_k=(\phi_y(a_0), \phi_y(a_1), \phi_y(a_2)\cdots, 1, \phi_y(a_{k+1})\cdots, \phi_y(a_n), \cdots)$,
where $\phi_y(a_n)=0$ or 1 and $y_0, y_1, \cdots, y_k\cdots, \epsilon I$. Then y_i and y_j $(i\neq j)$ have distinct proper binary expansion. If $y_1\neq y_2$, then $\phi_{y_1}\neq\phi_{y_2}$. Thus ϕ is injective and $\#(I)\leq\#(\{0,1\})$, that is, $c\leq 2^{\aleph_0}$. By the cantor-Bernstein-schröder theorem $c=2^{\aleph_0}$.

We can prove Lemma 5 from theorem 5.

Lemma 5 : Let M be a set of cardinal number m and $B(M)$ be the set whose elements are all possible subsets of the set M. Then $B(M)$ has greater cardinal number than the cardinal number m of the initial set M, i.e., $m<2^m$. [1].

From Theorem 5 we deduce that $\aleph_0<2^{\aleph_0}=c<2^{(2^{\aleph_0})}<2^{2^{(2^{\aleph_0})}}<\cdots\cdots$.

Here, does there exist a cardinal number α between $\aleph_0$ and $2^{\aleph_0}$? There does not exist a cardinal number α such that $\aleph_0<\alpha<2^{\aleph_0}$ (THE CONTINUUM HYPOTHESIS). This statement cannot be proved or disproved. In fact it was proved by K. Gödel in 1938 that by accepting The Continuum Hypothesis as a true statement one does not introduce a contradiction. In 1964 P. J. Cohen proved that no contradiction is introduced if the negation of The Continuum Hypothesis is accepted as a true statement. The sets which most mathematicians use in their work are of cardinality not exceeding 2^c, and most of them turn out to be either of cardinality $\aleph_0$ or $\mathbf{c}$. [4]

From the above statement we can imply as follows: There is no greatest cardinal number and $\aleph_0$ is the smallest cardinality of infinite set.

Lemma 6 : (1) $\aleph_0^{\aleph_0}=2^{\aleph_0}=\mathbf{c}$.

(2) $\mathbf{c}^{\aleph_0}=2^{\aleph_0}$.

(3) $\aleph_0^n=\aleph_0$.

(4) $\mathbf{c}^n=\mathbf{c}$, for every n in N.

Proof:(1) It is clear that $\{0,1\}^N\subset N^N\Rightarrow 2^{\aleph_0}\leqq\aleph_0^{\aleph_0}$. Let $f\epsilon N^N: N\longrightarrow N$ is the mapping. Then $\#(N^N)\leq\#(B(N\times N))=\#(2^N)$, since N^N is a subset of $B(N\times N)$. Therefore $\aleph_0^{\aleph_0}\leq 2^{\aleph_0}$, and $\aleph_0^{\aleph_0}=2^{\aleph_0}=\mathbf{c}$.

(2) clearly $\{0,1\}^N\subset R^N\Rightarrow 2^{\aleph_0}\leq\mathbf{c}^{\aleph_0}$, since $\mathbf{c}^{\aleph_0}\leq\aleph_0^{\aleph_0\aleph_0}=\aleph_0^{\aleph_0}=2^{\aleph_0}$, then $\mathbf{c}^{\aleph_0}\leq 2^{\aleph}$, and whence $\mathbf{c}^{\aleph_0}=2^{\aleph_0}$.

(3) Let $\#(N^{\{1\}})=\#(N)^{\#(\{1\})}=\aleph_0^1$. There exists a bijection $\Phi: N^{\{1\}}\longrightarrow N$ defined by $\Phi_f(1)=n$ for every $f\epsilon N^{\{1\}}$ and $n\epsilon N$, hence $\aleph_0^1=\aleph_0$. Let $\#(N^{\{1,2\}})=\#(N)^{\#(\{1,2\})}=\aleph_0^2$, $\exists\rho: N^{\{1,2\}}\longrightarrow N$, a bijection defined by $\rho_g(1)=n+1$, $\rho_g(2)=n+2$ for every $g\epsilon N^{\{1,2\}}$ and $n\epsilon N$, hence $\aleph_0^2=\aleph_0$. If we assume the above statement true for $n\epsilon N\Rightarrow\aleph_0^n=\aleph_0$, then we have $\aleph_0^{n+1}=\aleph_0^n\cdot\aleph_0=\aleph_0\cdot\aleph_0=\aleph_0$, and whence $\aleph_0^n=\aleph_0$.

(4) is analogous to (3)

V. The cardinality of the set of function and subset.

Theorem 6 : (1) Let $B_k(A)$ be the set of all finite subset of A, then $\#(B_k(N))=\aleph_0$

(2) Let $f: R\rightarrow R$ be a continous function, then $\#(\{f\epsilon R^R|f: R\rightarrow R: \text{continuous}\})=\mathbf{c}$.

Proof:(1) Let $B_k(N)=\{A\epsilon B(N)\,|\,\#(A)=K\}$ and $J_k=\{\{n+1,n+2,\cdots\cdots n+k\}\,|\,n\epsilon N\}\subset B_k(N)$ for every $k\epsilon N$.

Then $\#(J_k)=\aleph_0$, and thus $\aleph_0\leq\#(B_k(N))$ Let the mapping $f:\{1,2,\cdots,k\}\rightarrow N$ be an injection. Then $\exists$ an injection $\rho: B_k(N)\longrightarrow N^{\{1,2,3,\cdots,k\}}$ defined by $\rho_A(k)=f_A(k)$ for all $k\epsilon\{1,2,\cdots,k\}$ and $f_A\epsilon N^{\{1,2,3,\cdots,k\}}$, $\#(A)=K$, since f_A is injective. Thus $\#(B_k(N))\leq\#(N^{\{1,2,\cdots,k\}})=\aleph_0^k=\aleph_0$, therefore $\#(B_k(N))=\aleph_0$.

(2) Let $T=\{f\epsilon R^R|f: R\rightarrow R: \text{continuous}\}$ and Q be the set of rational numbers, then for each $f\epsilon T$, $\exists$ a function $f_0: Q\longrightarrow R$ such that $f_0(x)=f(x)$ for all $x\epsilon Q$. Let a mapping $\mu: T\longrightarrow R^Q$ defined by $\mu_f(x)=f_0(x)$ for all $x\epsilon Q$. Then μ_f is injective. If $f\neq g$ are in T, then $\exists\, x_0\epsilon R$ s.t. $f(x_0)\neq g(x_0)$, i.e., $f(x_0)<g(x_0)$. By Continuity, $\exists\,\delta>0$, s.t., $f(x)<g(x)$ for all $x\epsilon(x_0-\delta, x_0+\delta)$. Since we can find a rational number $x\epsilon(x_0-\delta, x_0+\delta)$, then $f_0(x)<g_0(x)$, whence $f_0\neq g_0$.

Therefore μ is injective, and $\#(T)\leq\#(R^Q)=\mathbf{c}^{\aleph_0}=\mathbf{c}$. Next, since $\{0,1\}^N\subset R^N\subset R^R$, then $\mathbf{c}^{\aleph_0}=2^{\aleph_0}=\mathbf{c}\leq\#(T)$ By the Cantor-Bernstein-Schröder theorem $\#(T)=\mathbf{c}$.

References

[1] A.N. Kolomogorov and S.V. Fomin, Elements of the Theory of Functional Analysis, 9—11 (1957).

[2] Seymour Lipschutz, Theory and Problems of Set Theory Related Topics, 134—148(1964).

[3] Charles.C. Pinter, Set Theory, 150—155 (1971)

[4] S. Swierczkowski, Sets and Numbers 25—60 (1972)

[5] John E. Yarnelle, An Introduction to Transfinite Mathematics, 1—65 (1964).