다구찌의 2단계 최적화가 가능한 모형에 관한 연구

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<요 약>

파라미터 설계는 평균이차순합수 즉, 목표치로 부터 평균제곱편차를 최소화하는 설계변수의 범위를 찾아서 한다. 그러나, 실제로 설계변수의 최적 배치를 찾기 위해서 다구찌는 SN 비를 최대로 하는 배치를 찾아서 한다. 일반적으로 그는 이 두가지 최적화문제에 관한 연관성을 주지 않으나, Leon등은 PerMIA의 개념을 사용하여 어떤 모형에서 S/N 비를 최대로 하는 것이 평균이차순합수를 최소로 하는 것임을 보였다. 본 논문에서는 다구찌의 2단계 최적화의 가능성에 관한 정의를 내리고, 다구찌의 2단계 최적화가 가능한 경우에 이차순합수가 최소화된다는 것을 보였다. 또한 다구찌의 2단계 최적화의 가능성에 관한 필요충분조건을 제시했다. 이 결과로 Leon등의 모델에 관한 제약조건은 약화되었다.

A Study on the Model that Taguchi’s Two-Step Optimization is possible

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<Abstract>
In parameter design, to find the settings of product or process design parameters that minimize average quadratic loss that is, the average squared deviation of the response from its target value. Yet, in practice, to choose the settings design parameter he maximize a set of measures called signal-to-noise (SN) ratios. Taguchi gives no connection between these two optimization problems, but using the concept of a Performance Measure Independent of Adjustment (PerMIA), Leon, Shoemaker, and Kackar(1987) shows that if the transfer function is given a strictly monotone function of each component of a for \( d \), then use of SN ratio for a specific target being best in two-step procedure leads to minimization of average quadratic loss. In this study the possibility of Taguchi’s two-step optimization is define and it will be shown that the expected loss function is minimized if Taguchi’s two-step optimization is possible. Furthermore, one necessary and sufficient condition for the possibility of Taguchi’s two-step optimization will be introduced. From this result, the constraints for underlying methods of Leon et al. can be weaken.

1. Introduction

Recent years have seen an upsurge of interest in the role of statistical ideas and methods in improving the quality and productivity of industrial processes and products. Among other things great interest has been stimulated by Japanese quality expert Genichi Taguchi and his co-workers in the application of statistically designed experiments to industrial design.

According to Taguchi(1988), “Quality is the loss imparted to the society from the time a product is shipped”. The concept of societal loss gives us a new way to think about quality improvement. Performance characteristics of a product are the primary quality characteristics that determine the product’s performance in satisfying customer’s requirements. In order to determine the degree of satisfaction with a performance characteristic, the ideal state of the performance characteristic from the customer’s viewpoint must be known. The ideal state is called the target value. The variation of a performance characteristic about its target value is referred to as performance variation. The smaller the performance variation about the target value, the better is quality. Therefore, a high quality product performs near the target value consistently throughout the product’s life span and under all different operating conditions.

Off-line quality control methods are quality and cost control activities conducted at the product and process design stages to improve product manufacturability and
reliability, and to reduce product development and lifetime costs.

Parameter design(1990) is an off-line quality control method, popularized by Taguchi, for designing products and manufacturing processes that achieve the robustness against noise and reduced cost. Noise is hard-to-control variability affecting performance. Parameter design, Taguchi style, involve experiment design techniques utilizing orthogonal arrys and the signal-to-noise(SN) ratios. Orthogonal arrays are generalized Graeco-Latin squares. All common fractional factorial designs are orthogonal arrays(1985).

In parameter design, Taguchi’s stated objective is to find the settings of product or process design parameters that minimize average quadratic loss—that is, the average squared deviation of the response from its target value. Yet, in practice, to choose the settings design parameter he maximizes a set of measures called signal-to-noise(SN) ratios. In Chapter, the Taguchi approach to parameter design is introduced. In general, Taguchi gives no connection between these two optimization problems, but using the concept of a Performance Measure Indepentent of Adjustment(PerMIA), Leon, Shoemaker, and Kackar(1987) shows that if the transfer function is given by $Y = \mu(d, a)\varepsilon(N, d)$, where $E_N(Y) = \mu(d, a)$ is a strictly monotone function of each component of $a$ for each $d$, then use of SN ratio for a specific target being best in two-step procedure leads to minimization of average quadratic loss.

In this study the possibility of Taguchi’s two-step optimization is defined and it will be shown that the expected loss function is minimized if Taguchi’s two-step optimization is possible. Furthermore, one necessary and sufficient condition for the possibility of Taguchi’s two-step optimization will be introducted. From this result, the constraints for underlying methods of Leon et al. can be weaken.

2. Explanation of Taguchi’s SN ratio
by Leon, Shoemaker and Kackar

For the analysis of designed experiments, Taguchi(1985) uses performance measures that he calls signal-to-noise(SN) ratios. The performance measure in $SN = 10 \log_{10}(\bar{y}^2/s^2)$ was to be used in preference to the standard deviation
for the problem of achieving, for some characteristic output $Y$, the smallest expected loss about an target value. In explaining the use of $SN$ ratio, Phadke said why we worked in terms of the $SN$ ratio rather than the standard deviation (Box(1988)):

Frequently, as the mean decreases the standard deviation also decreases and vice versa. In such cases, if we work in terms of the standard deviation, the optimization cannot be done in two steps i.e. we cannot minimize the standard deviation first and then bring the mean on target. Among many application, Taguchi has empirically found that two stage optimization procedure involving the $SN$ ratio indeed gives the parameter level combination where the standard deviation is minimum while keeping the mean on target. This implies that the engineering systems behave in such a way that manipulatable production factors can be divided into three categories:

(1) control factors, which affect process variability as measured by the $SN$ ratio,
(2) signal factors, which do not influence (or have negligible effect on) the $SN$ ratio but have a significant effect on the mean, and
(3) factors which do not affect the $SN$ ratio or the process mean.

The idea is carried further in the paper by Leon et al.(1987). In their terminology, given that the proceeding division of the factors can be achieved, the $SN$ ratio would be an example of a Performance Measure Independent of Adjustment(PerMIA). The signal factors would be called adjustment factors, and the control and signal factors together would be design factors.

For a given setting of design parameters, $\theta$, noise $N$ produces a characteristic output $Y$. The output is determined by the transfer function, $f(N;\theta)$. The noise assumed to be random; hence the output will be random. A loss is incurred if the output differs from a target $\tau$, which represents the ideal output.

The goal of parameter design is to choose the setting of the design parameter $\theta$ that minimizes average loss caused by deviations of the output from target. This average loss is given by $R(\theta) = E_N\{L(Y)\}$, where $L$ is a loss function.

Taguchi called this problem the static parameter design problem, because the target is fixed (Leon et al(1987)). He recommended that loss be measured using the quadratic loss function $L(Y) = k(Y-\tau)^2$. 
To solve the problem just described, Taguchi generally divided the design parameter into two groups, \( e = (d, a) \), and used the following two-step procedure (Leon et al.(1987)):

**Two-step procedure**:

Step 1. Find the setting \( d = d^* \) that maximizes the SN ratio.

Step 2. Adjust \( a \) to \( a^* \) while \( d \) is fixed at \( d^* \).

The division of design parameters into two groups is motivated by the idea that the parameters \( a \) are fine-tuning adjustments that can be optimized after the main design parameters \( d \) are fixed. Taguchi claimed that this two-step approach has the advantage that, once a design parameter \( (d^*, a^*) \) is established, certain changes to product or process the adjustment parameter, \( a \). The initial setting of \( d = d^* \) remains optimal.

In general, Taguchi gave no justification for the use of the SN ratios and no explanation of why the two-step procedure that he recommended will minimize average loss (Leon et al.(1987)). But using the concept of a Performance Measure Independent of Adjustment (PerMIA), Leon et al.(1987) shows that if the transfer function is given by

\[
Y = \mu(d, a)\varepsilon(N, d)
\]

(2.1)

where \( E_N(Y) = \mu(d, a) \) is a strictly monotone function of each component of \( a \) for each \( d \), then use of \( SN = 10 \log_{10}(\bar{y}^2/s^2) \) in a two-step procedure leads to minimization of average quadratic loss. (Note that \( E_N(Y) = \mu(d, a) \) implied that \( E_N(\varepsilon(n, d)) = 1 \).

But the constraints of model \( Y = \mu(d, a)\varepsilon(N, d) \) can be weaken. In the following section it is shown that the constraints for underlying model of Leon et al. can be weaken.
3. The model that Taguchi's two-step optimization is possible

Let $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$ represent design parameters, and $N$ represent the noise factors in parameter design experiment. The noise assumed to be random. In the case that design parameters $\theta$ is decomposed into $\theta = (d, a)$ such that $\text{Var}_N(Y)/E^2_N(Y)$ is a function of only $d$,

Let

$$\sigma^2(d) = \text{Var}_N(Y)/E^2_N(Y), \quad \mu(\theta) = E_N(Y) \quad (3.1)$$

Also let Taguchi's two-step optimization define as follows (Leon et al.1987).

Taguchi's two-step optimization:

Step 1. Choose $d = d^*$ that minimize $\sigma^2(d)$.

Step 2. Adjust $a = a^*$ such that $\mu(d, a) = \tau/(1 + \sigma^2(d^*))$

So if Taguchi's two-step optimization is possible, from Step 1, $\theta$ is decomposed into $\theta = (d, a)$ such that $\text{Var}_N(Y)/E^2_N(Y)$ is a function of only $d$ and from Step 2, there exists an $a = a^*$ such that $\mu(d, a) = \tau/(1 + \sigma^2(d^*))$, for $d = d^*$ that minimizes $\sigma^2(d)$. Therefore in this study the possibility of Taguchi's two-step optimization is defined as follows.

DEFINITION.

Taguchi’s two-step optimization is possible if $\theta$ is decomposed into $\theta = (d, a)$ such that $\text{Var}_N(Y)/E^2_N(Y)$ is a function of only $d$, and there exists an $a = a^*$ such that $\mu(d, a) = \tau/(1 + \sigma^2(d^*))$, for $d = d^*$ that minimizes $\sigma^2(d)$. 
RESULT 1.

The expected loss function is minimized if Taguchi’s two-step optimization is possible.

PROOF.

The expected loss function is as follows.

\[
E_N(Y - \tau)^2 = E_N(Y - \mu + \mu - \tau)^2
\]
\[
= E_N(Y - \mu)^2 + (\mu - \tau)^2
\]
\[
= \mu^2(d, a) \sigma^2(d) + (\mu(d, a) - \tau)^2
\]
\[
= (1 + \sigma^2(d)) \left[ \mu(d, a) - \frac{\tau}{1 + \sigma^2(d)} \right]^2 + \frac{\sigma^2(d)}{1 + \sigma^2(d)} \tau^2
\]

Also \((1 + \sigma^2(d)) \left[ \mu(d, a) - \frac{\tau}{1 + \sigma^2(d)} \right]^2\) is non-negative and \(\frac{\sigma^2(d)}{1 + \sigma^2(d)} \tau^2\) is a strictly increasing function of \(\sigma^2(d)\). Therefore, \(E_N(Y - \tau)^2\) is minimized if there exists an \(a = a^*\) that minimized \(\sigma^2(d)\). This completes the proof.

From this results, if Taguchi’s two-step optimization is possible, Taguchi’s two-step optimization is an excellent approach. Let \(\varepsilon(N, a) = Y/\mu(a)\), Then \(E_N(\varepsilon(n, d)) = 1\). So every model depending upon \(a\), \(N\) can be expressed as \(Y = \mu(d, a)\varepsilon(N, d)\), where \(E_N(\varepsilon(n, d)) = 1\).

If Taguchi’s two-step optimization is possible, \(a\) is decomposed into \(a = (d, a)\) such that \(\text{Var}(\varepsilon(N, a)) (\text{Var}(Y)/E_N(Y))\) is independent of \(a\), i.e., a function of only \(d\). But there exists a function that satisfies the following properties.
COUNTEREXAMPLE.

There exists a non-negative function $\varepsilon(N, d, a)$ that satisfies the following properties,

1. $E_N\{\varepsilon(N, d, a)\} = 1$ for any $\varepsilon$.
2. $Var_N\{\varepsilon(N, d, a)\}$ is independent of $a$, but.
3. $\varepsilon(N, d, a)$ is a function of $a$. (where $N$ a random variable)

SOLUTION.

Let noise $N$ be a random variable with uniform distribution on $[0, 1]$. Then $E(N) = 1/2$ and $Var(N) = 1/12$

Let the number of the design parameters $\varepsilon$ be two, i.e., $d$ and $a$. Also let the parameter spaces of $d$ and $a$ be $D$, $A$, respectively. Let $D$, $A$ and $\varepsilon(N, d, a)$ define as follows.

$$D = \{d : 0 \leq d \leq 1\}, \quad A = \{a_1, a_2\}$$

$$\varepsilon(N, d, a) = \begin{cases} 
2(1-d)(N-1/2) + 1, & \text{if } 0 \leq N \leq 1, \; a = a_1, \; d \in D; \\
2(d-1)(N-1/2) + 1, & \text{if } 0 \leq N \leq 1, \; a = a_2, \; d \in D 
\end{cases}$$

Then $E_N\{\varepsilon(N, d, a)\} = 1$ and $Var_N\{\varepsilon(N, d, a)\} = (d-1)^2/3$ for any $\varepsilon$.

Therefore there exists a non-negative function $\varepsilon(N, d, a)$ such that $E_N\{\varepsilon(N, d, a)\} = 1$ for any $\varepsilon$ and $Var_N\{\varepsilon(N, d, a)\}$ is independent of $a$ and $\varepsilon(N, d, a)$ is a function of $a$.

In Taguchi's two-step optimization only expectation and variance of $Y$ is measured. Therefore, from the above fact, the model that Taguchi's two-step optimization is possible can not be specified, but the model that exactly explains only expectation and variable can be specified. In this case let the model be "exactly explicable up to expectation and variance of $Y"."
RESULT 2.

Taguchi’s two-step optimization is possible if and only if the model $Y = \mu(d, a)\varepsilon(N, d)$ such that $E_N(\varepsilon(N, d)) = 1$ is exactly explicable up to expectation and variance of $Y$ for any $\varepsilon = (d, a)$ and there exists an $a = a^*$ such that $\mu(d^*, a^*) = \tau/(1 + \sigma^2(d^*))$, for $d = d^*$ that minimizes $\sigma^2(d)$.

PROOF.

Since Taguchi’s two-step optimization is possible, $\varepsilon$ is decomposed into $\varepsilon = (d, a)$ such that $\text{Var}_N(Y)/E^2(Y)$ is a function of only $d$.

Let

$$f(N, d, a) = Y/\mu(d, a).$$

Then

$$Y = \mu(d, a)f(N, d, a)$$

Hence

$$E_N(\varepsilon(N, d, a)) = 1 \text{ and } \text{Var}_N(f(N, d, a)) = \text{Var}_N(Y)/E^2_N(Y) = \sigma^2(d).$$

Therefore $E_N(\varepsilon(N, d, a))$ and $\text{Var}_N(f(N, d, a))$ are independent of $a$. So let $\varepsilon(N, d) = f(N, d, a)$ for any $\varepsilon$. Then

The Model $Y = \mu(d, a)\varepsilon(N, d)$ such that $E_N(\varepsilon(N, d)) = 1$ is exactly explicable up to expectation and variance of $Y$ for any $\varepsilon = (d, a)$.

Also for $d = d^*$ that minimizes $\sigma^2(d)$, there exists an $a = a^*$ such that $\mu(d^*, a^*) = \tau/(1 + \sigma^2(d^*))$ by the definition of the possibility of Taguchi’s two-step optimization. Conversely, since the model $Y = \mu(d, a)\varepsilon(N, d)$ such that $E_N(\varepsilon(N, d)) = 1$ is exactly explicable up to the expectation and variance of $Y$ for any $\varepsilon$,

$$E_N(Y) = \mu(d, a)$$

$$\text{Var}_N(Y) = \text{Var}(\mu(d, a)\varepsilon(N, d))$$
\[ = \mu^2(d, a) \text{Var}_N(\varepsilon(N, d)) \text{for any } \varepsilon. \quad (3.4) \]

Therefore \( \text{Var}_N(Y)/E_{N}^2(Y) = \text{Var}_N(\varepsilon(N, d)) \) is a function of only \( d \). So this completes the proof.

**COROLLARY 1.**

If the model is given by \( Y = \mu(d, a)\varepsilon(N, d) \), where \( E_N(\varepsilon(N, d)) = 1 \), and there exists an \( a = a^* \) such that \( \mu(d^*, a^*) = \tau/(1 + \sigma^2(d^*)) \), for \( d = d^* \) that minimizes \( \sigma^2(d) \), then Taguchi’s two-step optimization is possible.

**COROLLARY 2. (Leon, Shoemaker, and Kackar (1987))**

If the model is given by \( Y = \mu(d, a)\varepsilon(N, d) \) where \( E_N(Y) = \mu(d, a) \) is a strictly monotone function of each component of \( a \) for each \( d \), then use of \( SN = 10 \log_{10}(\bar{y}^2/s^2) \) in a two-step procedure leads to minimization of average quadratic loss.

**PROOF.**

Since \( E_N(Y) = \mu(d, a) \) is a strictly monotone function of each component of \( a \) for each \( d \), there exists an \( a = a^* \) such that \( \mu(d^*, a^*) = \tau/(1 + \sigma^2(d^*)) \), for \( d = d^* \) that minimizes \( \sigma^2(d) \). So from corollary 1, Taguchi’s two-step optimization is possible. Therefore, from the result 1, the expected loss function is minimized. This completes the proof. For example, if the model is given by

\[ Y = \mu(d, a) + \varepsilon(N, d) \quad (3.5) \]

with \( E_N(\varepsilon(N, d)) = 0 \), then \( \text{Var}_N(Y)/E_{N}^2(Y) = \text{Var}_N(\varepsilon(N, d))/\mu^2(d, a) \).

So \( \text{Var}_N(Y)/E_{N}^2(Y) \) is not a function of only \( d \). Therefore Taguchi’s two-step optimization is not possible by the definition of Taguchi’s two-step
optimization. But the expected loss function is as follows.

\[ E_N(Y - \tau)^2 = E_N(Y - \mu + \mu - \tau)^2 \]
\[ = E_N(Y - \mu)^2 + (\mu - \tau)^2 = \text{Var}_N(Y) + (\mu - \tau)^2 \quad (3.6) \]

Therefore if there is an \( a = a^* \) such that \( \mu(d^*, a^*) = \tau \) for \( d = d^* \) that minimizes \( \text{Var}_N(Y) = \text{Var}_N(\epsilon(N, d)) \), then the following two-step optimization is possible and minimizes the expected loss function.

Two-Step Optimization:

Step 1. Choose \( d = d^* \) that minimizes \( \text{Var}_N(Y) \).
Step 2. Adjust \( a = a^* \) such that \( \mu(d^*, a^*) = \tau \)

Also if the model is given by

\[ Y = \mu(\theta) + \epsilon \quad (3.7) \]

with \( E(\epsilon) = 0 \) and \( \text{Var}(\epsilon) = \sigma^2 \) independent of \( \theta \), then

\[ E_N(Y - \tau)^2 = E_N(Y - \mu + \mu - \tau)^2 \]
\[ = E_N(Y - \mu)^2 + (\mu - \tau)^2 = \sigma^2 + (\mu(\theta) - \tau)^2 \quad (3.8) \]

Therefore if there is a \( \theta = \theta^* \) such that \( \mu(\theta^*) = \tau \), then the expected loss function is minimized. Of course, Taguchi's two-step optimization is not possible.

The model includes the linear mathematical models in experiment design. In this case quality problems involving noise(one of the important concept of Taguchi's parameter design) is excluded from the parameter design.
4. Conclusion

As seen in this study, if Taguchi’s two-step optimization is possible, the expected loss function is minimized. Therefore, this optimization is an excellent approach. Furthermore, the division of design parameters into two groups has the advantage that, once an optimal setting is established, certain changes to product or process requirements can be accommodated by changing only the setting of adjustment parameter. The initial setting of Step 1 in Taguchi’s two-step optimization remains optimal.

But this method has several shortcomings. First, the existence of special design parameters, called adjustment parameters, is assumed. In parameter design, adjustment parameters are used to force the mean to be close to the target value. But if adjustment parameters do not exist, Taguchi’s two-step optimization is not possible. Second there may be cases where adjustment parameter cannot be easily manipulated, though they exist. In other words, in cases where there are no adjustment parameters, or where adjustment parameters are hard to manipulate, Taguchi’s two-step optimization fails.

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